Fibonacci Numbers and Binet's Formula using Generating Functions/Infinite Sums

A *sequence* is a list of numbers that never ends (e.g. $1, 3, 5, 7, 9, 13, \ldots$) The *Fibonacci Sequence* is an exciting sequence of numbers we will talk about today. The first few numbers are:

f_0	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	...
0	1	1	2	3	5	8	13	21	34	...

The rule that makes the Fibonacci Sequence is "the next number is the sum of the previous two". This kind of rule is sometimes called a *recurrence relation.* Mathematically, this is written as:

$$f_n = f_{n-1} + f_{n-2}$$

There is an explicit formula for the n-th Fibonacci number known as Binet's formula:

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

In the rest of this note, we will explain how this works by using a really powerful idea called "*generating functions*" which let us attack these problems. Generating functions involve using algebra to solve *infinite sums.* Before we jump into Fibonacci, we will start with some warm up problems to get the hang of it. (Note that infinite sums can be slippery...not all infinite sequences can be summed. We won't worry about these technicalities here!) (Note also the solutions to the exercises are included on the next page)

Exercise 0: $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \ldots = ?$

Exercise 1: $1 + x + x^2 + x^3 + x^4 + \ldots = ?$

Exercise 2: $1 + 2x + 4x^2 + 8x^3 + 16x^4 + \ldots = ?$

Exercise 3: $1 + 2x + 3x^2 + 4x^3 + 5x^4 + \ldots = ?$

Generating Functions

For any sequence of numbers, there is a <u>*generating function*</u> associated with that sequence. (By a function, I mean an expression that depends on x.) The rule for the generating function is to multiply each term of the sequence by x^n, and finally do the infinite sum of all these terms. This is written mathematically as:

Definition: The generating function for the sequence $a_0, a_1, a_2, a_3, \ldots$ is the infinite sum:

$$S = a_0 \cdot x^0 + a_1 \cdot x^1 + a_2 \cdot x^2 + a_3 \cdot x^3 + \ldots$$

Here are some examples. We already did the work for these as exercises on the previous page!

Example 1: The generating function for the sequence $1, 1, 1, 1, 1 \ldots$ is $S = \frac{1}{1-x}$.

We must evaluate the infinite sum $S = 1 \cdot x^0 + 1 \cdot x^1 + 1 \cdot x^2 + 1 \cdot x^3 + \ldots$. We use the "shifting trick":

$$\begin{array}{rcccc}
S = & 1+ & x+ & x^2+ & x^3 + \ldots \\
- \quad xS = & & x+ & x^2+ & x^3 + \ldots \\
\hline
\therefore (1-x)\, S = & 1+ & 0+ & 0+ & 0 + \ldots
\end{array}$$

$$\therefore S = \frac{1}{1-x}$$

Example 2: The generating function for the sequence $1, 2, 4, 8, 16 \ldots$ is $S = \frac{1}{1-2x}$.

We must evaluate the infinite sum $S = 1 \cdot x^0 + 2 \cdot x^1 + 4 \cdot x^2 + 8 \cdot x^3 + \ldots$.

Notice that we can rewrite each term as a power of $2x$, namely $S = 1 \cdot x^0 + (2x)^1 + (2x)^2 + (2x)^3 + \ldots$. Now we can use the same shifting trick as before (Alternatively, you can "plug in $2x$ into Example 1):

$$\begin{array}{rccc}
S = & 1+ & (2x)+ & (2x)^2 + \ldots \\
- \quad 2xS = & & 2x+ & (2x)^2 + \ldots \\
\hline
\therefore (1-2x)\, S = & 1+ & 0+ & 0 + \ldots
\end{array}$$

$$\therefore S = \frac{1}{1-2x}$$

Example 3: The generating function for the sequence $1, 2, 3, 4, 5, 6 \ldots$ is $S = \frac{1}{(1-x)^2}$.

We must evaluate the infinite sum $S = 1 \cdot x^0 + 2 \cdot x^1 + 3 \cdot x^2 + 4 \cdot x^3 + \ldots$. We can use the same shifting trick to reduce this to the sum we already computed in example 1:

$$\begin{array}{rcccc}
S = & 1+ & 2x+ & 3x^2+ & 4x^3 + \ldots \\
- \quad xS = & & x+ & 2x^2+ & 3x^3 + \ldots \\
\hline
\therefore (1-x)\, S = & 1+ & x+ & x^2+ & x^3 + \ldots \\
= & \dfrac{1}{1-x} & & & \text{from Example 1}
\end{array}$$

$$\therefore S = \frac{1}{(1-x)^2}$$

Fact 1: The generating function for the Fibonacci sequence $0, 1, 1, 2, 3, 5, 8 \ldots$ is $S = \frac{x}{1-x-x^2}$.

We must evaluate the infinite sum $S = 0 \cdot x^0 + 1 \cdot x^1 + 1 \cdot x^2 + 2 \cdot x^3 + 3 \cdot x^4 \ldots$. Since the recurrence relation for the Fibonacci numbers involves the last *two* numbers, we must use a "double" shifting trick:

$$\begin{array}{rcccccc}
S = & 0+ & x+ & x^2+ & 2x^3+ & 3x^4+ & 5x^5 + \ldots \\
- \quad xS = & & 0+ & x^2+ & x^3+ & 2x^4+ & 3x^5 + \ldots \\
- \quad x^2S = & & & 0+ & x^3+ & x^4+ & 2x^5 + \ldots \\
\hline
\therefore \left(1-x-x^2\right) S = & 0+ & x+ & 0+ & 0+ & 0+ & 0 + \ldots
\end{array}$$

$$\therefore S = \frac{x}{(1-x-x^2)}$$

The reason all the 0's appear is from the recurrence relation for the Fibonacci numbers; in other words the fact that the next Fibonacci number is the sum of the previous two.

The Golden Ratio

Define two numbers φ and β to be the roots of the quadratic equation $x^2 - x - 1$. (This quadratic equation appeared "in reverse" in the denominator for the generating function of the Fibonacci numbers).

By the quadratic equation, these are:

$$\varphi = \frac{1+\sqrt{5}}{2} \approx 1.6180339887\ldots$$
$$\beta = \frac{1-\sqrt{5}}{2} \approx -0.6180339887\ldots$$

The number φ is called the *Golden Ratio* and has a number of exciting properties (go see the Wikipedia page for more info!). Both φ and β are intimately related to the Fibonacci sequence because they appear in the generating function!

Fact 2: $1 - x - x^2 = (1-\varphi x)(1-\beta x)$

You can check this fact by expanding it out using "FOIL" and the fact that $\varphi\beta = -1$ and $\varphi + \beta = 1$. Try it here!

Fact 3: Let f_n be the n-th Fibonacci number. Then we have the explicit formula: $f_n = \frac{1}{\sqrt{5}}(\varphi^n - \beta^n)$

The trick we use to get this result is to rewrite the generating function for the Fibonacci numbers $S = \frac{x}{(1-x-x^2)}$ we found in Fact 1 using the factorization $1 - x - x^2 = (1-\varphi x)(1-\beta x)$ we found in Fact 2. By some algebraic manipulations, (This is called the *partial fractions* trick!) we find that:

$$\frac{x}{1-x-x^2} = \frac{1}{\sqrt{5}}\frac{1}{1-\varphi x} - \frac{1}{\sqrt{5}}\frac{1}{1-\beta x}$$

Now we recognize that $\frac{1}{1-\varphi x}$ and $\frac{1}{1-\beta x}$ look a lot like the answer from Example 2. These are infinite sums!

$$\frac{1}{1-\beta x} = 1 + \beta x + \beta^2 x^2 + \beta^3 x^3 + \ldots$$
$$\frac{1}{1-\varphi x} = 1 + \varphi x + \varphi^2 x^2 + \varphi^3 x^3 + \ldots$$

Putting this all together gives:

$$\begin{aligned}
& f_0 + f_1 x + f_2 x^2 + f_3 x^3 + \ldots \\
= & \frac{x}{1-x-x^2} \\
= & \frac{1}{\sqrt{5}}\frac{1}{1-\varphi x} - \frac{1}{\sqrt{5}}\frac{1}{1-\beta x} \\
= & \frac{1}{\sqrt{5}}\left(1 + \varphi x + \varphi^2 x^2 + \varphi^3 x^3 + \ldots\right) - \frac{1}{\sqrt{5}}\left(1 + \beta x + \beta^2 x^2 + \beta^3 x^3 + \ldots\right) \\
= & 0 + \frac{1}{\sqrt{5}}(\varphi - \beta)x + \frac{1}{\sqrt{5}}\left(\varphi^2 - \beta^2\right)x + \frac{1}{\sqrt{5}}\left(\varphi^3 - \beta^3\right)x^3 + \ldots
\end{aligned}$$

Since these two "polynomials" are equal for every value of x, each term must individual by equal. Reading off the coefficients of the power x^n gives the final result,

$$f_n = \frac{1}{\sqrt{5}}(\varphi^n - \beta^n) = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

. This is Binet's formula!

Converting Miles to Kilometers

Factoid 1: Fibonacci numbers can be used to convert miles to kilometers by: $f_n\,\text{km} \approx f_{n-1}\,\text{mi}$

The secret of this factoid is an amazing coincidence between the numerical value of φ and the number of kilometers in a mile, and the fact that $|\beta| < 1$. Firstly, notice that:

$$\begin{aligned} \varphi &= 1.6180\ldots \\ \frac{1\text{mi}}{1\text{km}} &= 1.6093\ldots \end{aligned}$$

Because these two values are close, the approximation 1 mi $\approx \varphi\,$km is pretty good (to about 1%). Now notice that since $\beta < 1$, that β^n is really small as n gets larger; $\beta^n \approx 0$. So we have some more approximations:

$$\begin{aligned} f_n &= \frac{1}{\sqrt{5}}\left(\varphi^n - \beta^n\right) \approx \frac{1}{\sqrt{5}}\varphi^n \\ f_{n-1} &= \frac{1}{\sqrt{5}}\left(\varphi^{n-1} - \beta^{n-1}\right) \approx \frac{1}{\sqrt{5}}\varphi^{n-1} \\ \therefore f_n &\approx \varphi f_{n-1} \end{aligned}$$

Along with 1mi $\approx \varphi$km, this means that f_{n-1}km $\approx f_n$mi. This works best if n is not-too-small, because when n is large, our approximation that $\beta^n \approx 0$ becomes more accurate. $n = 5$ is already quite a good approximation ($\beta^4 \approx 0.0002$). The first couple listed for you, starting at $n = 5$:

$$\begin{aligned} 3\text{ mi} &\approx 5\,\text{km} \\ 5\text{ mi} &\approx 8\,\text{km} \\ 8\,\text{mi} &\approx 13\text{ km} \end{aligned}$$

If you want to covert numbers not on this list, you can bootstrap from the above approximations. For example, starting from 5mi $\approx$ 8km, you can do:

$$100\text{mi} = 20 \cdot 5\text{mi} \approx 20 \cdot 8\text{km} = 160\text{km}$$

If you found this interesting...

Here are some great websites that you can check out where you can get more!

Wikipedia links:

- Fibonacci number
- Golden ratio
- Recurrence relation
- Generating function

Other links:

- "Doodling in Math: Spirals, Fibonacci, and Being a Plant [1 of 3]" by Vi Hart. http://www.khanacademy.org/math/vi-hart/v/doodling-in-math–spirals–fibonacci–and-being-a-plant–1-of-3
- "Exercise - Write a Fibonacci Function" by Salman Khan (has a computer science flavor) https://www.youtube.com/watch?v=Bdbc1ZC-vhw